

# The Mean Value Theorem

## CHAPTER 3 SECTION 2

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### 1. In this section

If I drive a distance of 65 miles in one hour, we know that my average rate of speed during that hour was 65 mph. In fact, we can say more; my instantaneous speed at some time during that hour had to be 65. Why? Well, if I traveled less than 65 mph for the entire hour, I would not have gone so far, and if I had traveled faster than 65, I would have gone farther. If I traveled at 60 mph for half an hour and 70 mph for the other half hour, I still had to accelerate through 65 mph.

The Mean Value Theorem is a very important theoretical tool that expresses this concept precisely.

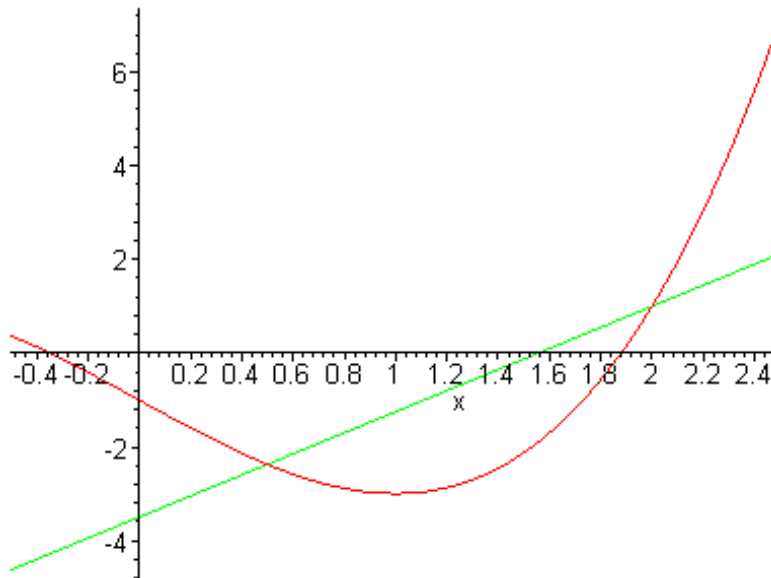
### 2. Introduction

When we first talked about the tangent line problem, we discussed the approximation of the slope of the tangent line by slopes of secant lines. The mean value theorem looks at the relationship between the slopes of secant and tangent lines from a different perspective. Geometrically, the mean value theorem tells us that, if we construct a secant line on the graph of any smooth function, say through the points  $(a, f(a))$  and  $(b, f(b))$ , where we assume  $a < b$ , then there is a number  $c$  between  $a$  and  $b$  so that the slope of the tangent line to the graph of  $f$  at the point  $(c, f(c))$  is equal to the slope of the secant line through  $(a, f(a))$  and  $(b, f(b))$ .

Let's look at a picture to illustrate this. We'll look at the graph of

$$f(x) = x^3 - 3x - 1$$

together with a secant line passing through the points  $(1/2, f(1/2))$  and  $(2, f(2))$ .



The slope of the secant line is

$$\frac{f(2) - f(1/2)}{2 - 1/2} = \frac{9}{4}.$$

The slope of the tangent line is, of course, given by the derivative of  $f$ ,

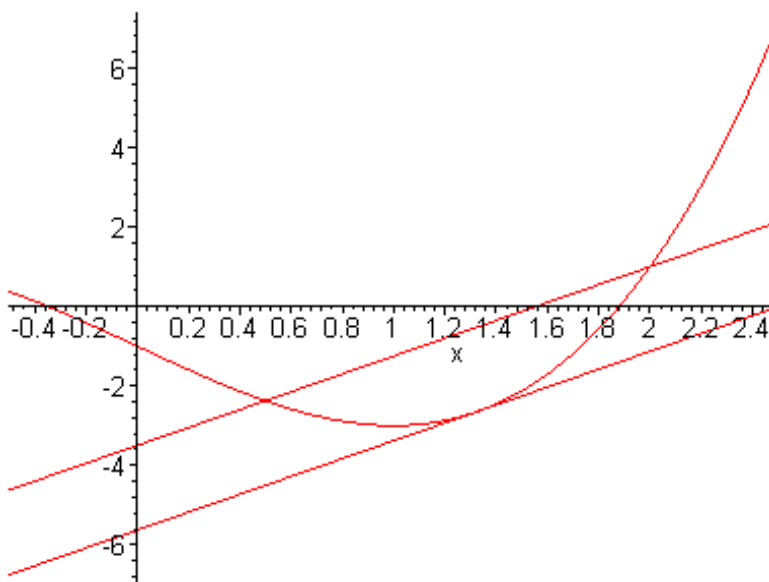
$$f'(x) = 3x^2 - 3$$

Looking at the graph, it appears that the tangent line would be parallel to the secant line at approximately  $x = 1.3$ . In this case, we can solve the necessary equation explicitly.

$$\begin{aligned} 3x^2 - 3 &= \frac{9}{4} \\ x^2 &= \frac{7}{4} \\ x &= \pm \frac{\sqrt{7}}{2} \approx \pm 1.323 \end{aligned}$$

As you can see, our visual estimate was very close.

Here is a picture showing the graph of the function together with the secant and tangent lines.



In general, it is not possible to find the point where the tangent line is parallel to the secant line exactly, but it is possible to approximate it, either graphically or using a numerical scheme (e.g. Newton's method).

Here is a formal statement of the mean value theorem (MVT).

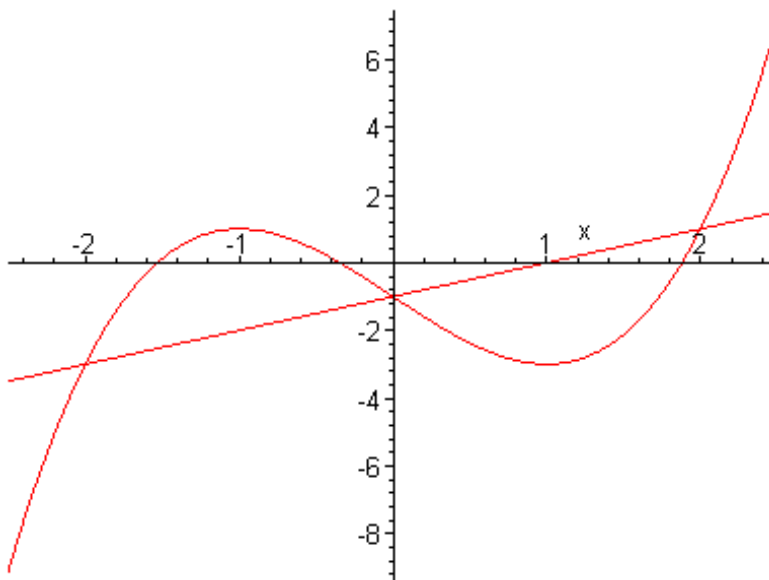
**Theorem 1 (Mean Value Theorem)** *If  $f$  is a continuous function on the closed interval  $[a, b]$ , differentiable at each point in the interior  $(a, b)$ , then there is a number  $c$ , with  $a < c < b$ , so that  $f(b) - f(a) = f'(c)(b - a)$ .*

An alternative to the geometric interpretation of the MVT is the following: For any smooth function on an interval  $[a, b]$ , there must be some point  $c$ ,  $a < c < b$ , so that the instantaneous rate of change of the function at  $c$  (i.e.  $f'(c)$ ) is equal to the average rate of change of the function over the interval  $[a, b]$  (which is simply  $(f(b) - f(a))/(b - a)$ .) So, for example, if I drive 60 miles in one hour, the MVT says that, at some time during that hour, I must have been driving at 60 mph. (It's not possible that I drove either over 60 mph or under 60 mph for the entire hour, nor is it possible that I somehow "skipped" from, say, 50 mph to 70 mph without passing through 60 mph.)

Note that the MVT does not say that the point at which the tangent line parallels the secant line

is unique. There may be more than one point. The MVT simply guarantees that at least one such point exists.

Here's an example. Let's use the same function as before, but look at the secant line passing through the points  $(-2, f(-2))$  and  $(2, f(2))$ .



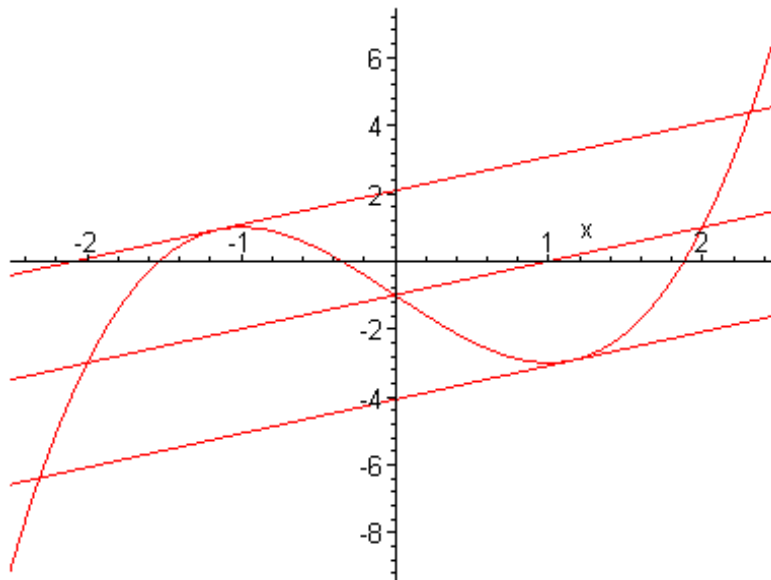
It appears in this case that the tangent line will be parallel to the secant line at approximately both  $x = -1.2$  and  $x = 1.2$ . Again, in this case, we can solve explicitly. The slope of the tangent line through the two points is

$$\frac{f(2) - f(-2)}{2 - (-2)} = \frac{1 - (-3)}{4} = 1.$$

Setting  $f'(x) = 1$  we find

$$\begin{aligned} 3x^2 - 3 &= 1 \\ x^2 &= \frac{4}{3} \\ x &= \pm \frac{2}{\sqrt{3}} \approx \pm 1.155 \end{aligned}$$

Here's a plot of the function with the secant line and both tangent lines.



A special case of the MVT, known as Rolle's Theorem, occurs when  $f(a) = f(b)$ . In this case, the average rate of change of the function over the interval  $[a, b]$  is 0, so the MVT guarantees that there will be a point on the graph at which the tangent line is horizontal; i.e. a point on the graph at which the derivative is 0.

So, for example, between any two roots of a differentiable function, there must be at least one point where derivative is 0. Note that, for the example we have been considering, the tangent line

is horizontal at  $x = -1$  and  $x = 1$ .

