Limits at Infinity: Asymptotes

Chapter 3 Section 5

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1. In this section

We have already discussed vertical asymptotes, which occur when a function has an infinite discontinuity (for example, $f(x) = 1/x$ at $x = 0$). We also want to be able to identify asymptotes "at infinity"; that is, lines $L(x) = mx + b$ which the function gets arbitrarily close to for large $|x|$. Such asymptotes are called horizontal if $m = 0$, oblique otherwise. So, for the function f to have the line $L(x)$ as a horizontal or oblique asymptote we must have either

$$\begin{aligned} \lim_{x\to\infty} |f(x) - L(x)| &= 0, \text{ or} \\ \lim_{x\to-\infty} |f(x) - L(x)| &= 0. \end{aligned}$$

2. Introduction

The best examples to illustrate horizontal and oblique asymptotes are rational functions. However, before proceeding, we need to recall the following facts regarding limits of quotients of polynomials. If p and q are polynomials, then:

If the degree of p is less than the degree of q,

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = 0.$$

If the degree of p is equal to the degree of q,

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \frac{a}{b},$$

where a and b are the leading coefficients of p and q, respectively. Finally, If the degree of p is greater than the degree of q,

$$\lim_{x\to\infty} \frac{p(x)}{q(x)}$$

does not exist. (Note: The same results hold at negative infinity.)

So, in the first two cases (degree of p less than or equal to the degree of q), the graph of $y = p(x)/q(x)$ will have a horizontal asymptote (at 0 or a/b, respectively).

It turns out that, if the degree of p is exactly one more than the degree of q, the graph will have an oblique asymptote. The slope will be the quotient a/b of leading coefficients, while the constant term will be c/b, where c is the coefficient of the second term of p.

Here are some examples:

The graph of

$$y = \frac{x^2 + x - 1}{x^4 + 1}$$

will have a horizontal asymptote of 0, since the degree of the denominator is greater than that of the numerator.

The graph of

$$y = \frac{x^3 - x^2 - 3x + 7}{2x^3 + x^2 + 12}$$

will have a horizontal asymptote at 1/2, the quotient of the coefficients of the leading terms, since the degree of the numerator and denominator are the same.

The graph of

$$y = \frac{x^5 + x^3 - 3x + 1}{x^4 + 5}$$

will have the line $y = x$ as oblique asymptote; the slope 1 is the quotient of the leading terms, and the coefficient of x^4 in the numerator is 0, which gives a constant term of 0 for the asymptote.

The graph of

$$y = \frac{x^4 - x^3 + x + 1}{x^2 + 1}$$

won't have any oblique asymptotes; the degree of the numerator is 2 more than the degree of the denominator, so the limit as $x \to \infty$ will not be finite.