# Improper Integrals

## Chapter 7 Section 8

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## 1. In this section

The fundamental theorem of calculus tells us that, if $f$ is a continuous function on an interval $[a, b]$ and $F'(x) = f(x)$ for $x \in [a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Slight modifications allow us to evaluate definite integrals on an interval where $f$ is piecewise continuous with jump or removeable discontinuities. It is our goal in this section to extend our approach to interpret integrals on unbounded intervals or on intervals on which $f$ has an infinite discontinuity (a vertical asymptote).

So for example, since

$$\frac{d}{dx}\left(\frac{-1}{x}\right) = \frac{1}{x^2},\ x \neq 0,$$

we will have

$$\int_a^b \frac{1}{x^2} dx = \left(-\frac{1}{b}\right) - \left(-\frac{1}{a}\right) = \frac{1}{a} - \frac{1}{b}$$

as long as $0 \notin [a, b]$ (that is, as long as both $a$ and $b$ are both positive or both negative). However, if $0 < a$, we see that

$$\lim_{b\to\infty} \int_a^b \frac{1}{x^2} dx = \lim_{b\to\infty} \left(\frac{1}{a} - \frac{1}{b}\right) = \frac{1}{a}.$$

We can interpret this result geometrically as representing the area under the graph of the curve $y = 1/x^2$ on the unbounded interval $[a, \infty]$, and in fact we will write

$$\int_a^\infty \frac{1}{x^2} dx = \frac{1}{a},\ a > 0,$$

and call this an improper integral.

On the other hand, the graph of $y = 1/x^2$ has a vertical asymptote at $x = 0$ (the $y-$ axis); and, for $0 < b$ we have

$$\lim_{a\to 0^+}\int_a^b \frac{1}{x^2}dx = \lim_{a\to 0^+}\left(\frac{1}{a}-\frac{1}{b}\right) = \infty.$$

We therefore write

$$\int_0^b \frac{1}{x^2}\,dx = \infty,\ b > 0.$$

This is a divergent improper integral. (The previous improper integral is called convergent.)

## 2. Definitions and examples

We will define two basic types of improper integrals. First, if $f$ has an infinite discontinuity at $a$ and $a < b$ then we will define

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx.$$

Similarly, if $f$ has an infinite discontinuity at $b$, we define

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx.$$

Next, we define

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

and

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

The integrals are convergent if the limits are finite, divergent otherwise. Various combinations of these can occur, which will be illustrated through some examples.

Consider

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

This is an improper integral, since $1/\sqrt{x}$ has a vertical asymptote at $x = 0$, the left endpoint of the interval of integration. So,

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}\,dx &= \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx \\ &= \lim_{a\to 0^+}\left(\frac{x^{1/2}}{1/2}\right)_a^1 \\ &= \lim_{a\to 0^+}\left(2\left(1-a^{1/2}\right)\right)\end{aligned}$$

$$= \quad 2.$$

So, this is a convergent improper integral. (By the way, since $1/\sqrt{x}$ is the inverse of $1/x^2$ for $x > 0$, the convergence (and values) of the integrals $\int_0^1 1/\sqrt{x}\,dx$ and $\int_1^\infty 1/x^2\,dx$ are related. Can you explain the relation geometrically?)

We have already seen that for $a > 0$, $\int_a^\infty 1/x^2\,dx = 1/a$. What can we say, in general, about the convergence of the improper integrals

$$\int_1^\infty \frac{1}{x^p}\,dx?$$

First, note that, if $p \leq 0$, the integrand $1/x^p$ does not even approach $0$ as $x \to \infty$, so we know that the integrals will diverge for those values of $p$. In general, for $p > 0$ but $p \neq 1$,

$$\begin{aligned}
\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty} \int_1^b x^{-p}\,dx \\
&= \lim_{b\to\infty} \left(\frac{x^{-p+1}}{-p+1}\right)_1^\infty \\
&= \lim_{b\to\infty} \left(\frac{1}{p-1}\left(1 - b^{1-p}\right)\right) \\
&= \begin{cases} \frac{1}{p-1}, & \text{if } p > 1 \\ \infty, & \text{if } p < 1. \end{cases}
\end{aligned}$$

For the special case $p = 1$,we have

$$\begin{aligned}
\int_1^\infty \frac{1}{x}\,dx &= \lim_{b\to\infty} (\ln x)_1^b \\
&= \lim_{b\to\infty} (\ln b - \ln 1) \\
&= \lim_{b\to\infty} (\ln b) = \infty.
\end{aligned}$$

So, for $a > 0$ the improper integral

$$\int_a^\infty \frac{1}{x^p}\,dx$$

converges for $p > 1$ and diverges for $p \leq 1$. A similar argument (which you should construct) reveals that, for $b > 0$, the improper integral

$$\int_0^b \frac{1}{x^p}\,dx$$

converges for $p < 1$ and diverges for $p \geq 1$. (Again, there is an inverse relationship between the functions $1/x^p$ and $1/x^{1/p}$ which can be used to understand these results.)

Here is an example illustrating how more than one improper integral can be combined in a single integral:

$$\int_{-1}^{1} \frac{1}{\sqrt[3]{x}}\, dx$$

Since the integrand has a singularity at $x = 0$, we must decompose this integral into the sum of two improper integrals to determine whether or not it converges:

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{\sqrt[3]{x}}\, dx &= \int_{-1}^{0} \frac{1}{\sqrt[3]{x}}\, dx + \int_{0}^{1} \frac{1}{\sqrt[3]{x}}\, dx \\
&= \lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{\sqrt[3]{x}}\, dx + \lim_{a\to 0^+} \int_{a}^{1} \frac{1}{\sqrt[3]{x}}\, dx \\
&= \lim_{b\to 0^-} \left(\frac{x^{2/3}}{2/3}\right)_{-1}^{b} + \lim_{a\to 0^+} \left(\frac{x^{2/3}}{2/3}\right)_{a}^{1} \\
&= \lim_{b\to 0^-} \left(\frac{3}{2}\left(b^{2/3} - 1\right)\right) + \lim_{a\to 0^+} \left(\frac{3}{2}\left(1 - a^{2/3}\right)\right) \\
&= -\frac{3}{2} + \frac{3}{2} = 0.
\end{aligned}$$

Now, we might have deduced that the value of this integral (if it exists at all) must be $0$ since the integrand is an odd function. However, we must determine whether or not the improper integral converges before using this type of consideration. To see this is so, consider the integral

$$\int_{-1}^{1} \frac{1}{x}\, dx.$$

While the integrand is odd, neither of the integrals

$$\int_{-1}^{0} \frac{1}{x}\, dx \text{ or } \int_{0}^{1} \frac{1}{x}\, dx$$

converge. So, the original integral diverges as well.

## 3. Using comparison to determine convergence properties

In the examples considered in the previous sections, we were always able to find an antiderivative to aid in determining the convergence or divergence of the improper integrals under consideration. However, there may be cases where this is impossible. For example, consider the improper integral

$$\int_{0}^{\infty} \exp\left(-x^2\right)\, dx.$$

We do not know an antiderivative for the integrand $\exp\left(-x^2\right)$. How might we determine whether or not the integral converges?

Since $0 \leq \exp\left(-x^2\right) \leq 1$ for all $x$ and, for $x \geq 1$, $\exp\left(-x^2\right) \leq \exp\left(-x\right)$, we will have

$$\begin{aligned} 0 &< \int_0^\infty \exp\left(-x^2\right)\, dx < \int_0^1 dx + \int_1^\infty \exp\left(-x\right)\, dx \\ &= 1 + \lim_{b\to\infty}\left(-\exp\left(-x\right)\right)_1^b \\ &= 1 + \frac{1}{e}. \end{aligned}$$

Therefore, the improper integral

$$\int_0^\infty \exp\left(-x^2\right)\, dx$$

is convergent. (In fact, a combination of this idea, coupled with numerical integration techniques, can provide quite satisfactory approximations of the value of the integral.)

Now consider the improper integral

$$\int_a^\infty \frac{1}{\ln x}\, dx$$

where $a > 1$. While we cannot determine an antiderivative for the integrand $1/\ln x$, we do know that

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0.$$

(If you do not recall this limit, apply L'Hospital's rule.) Since, for $x > 1$, $\ln x/x > 0$, this means that for $x$ sufficiently large, $\ln x/x < c$ for any positive constant $c$, so, in particular (with $c = 1$), we can find a number $M > 1$ so that, for all $x > M$,

$$\frac{1}{x} < \frac{1}{\ln x}.$$

Therefore,

$$\int_M^\infty \frac{1}{x}\, dx < \int_a^\infty \frac{1}{\ln x}\, dx.$$

But the smaller integral diverges; therefore, so does the larger.