An application of implicit differentiation

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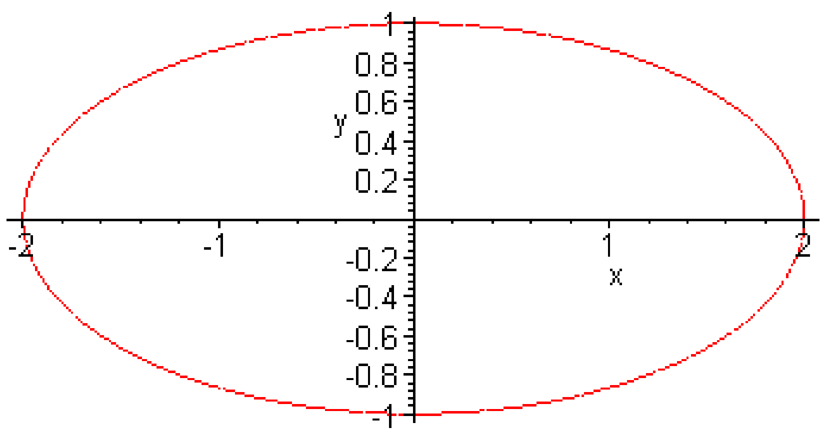
1. Problem statement

The equation of an ellipse with center at the origin $(0, 0)$, major axis along the $x-$ axis of length 4 and minor axis along the $y-$ axis of length 2 is

$$\frac{x^2}{4} + y^2 = 1.$$

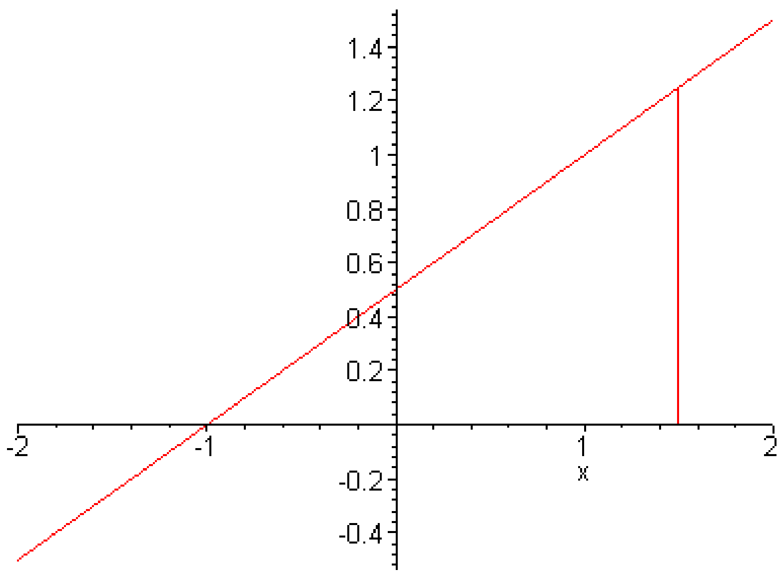
Here is a graph of the ellipse.



There are 4 points on this ellipse at which the tangent line intersects the $x-$ axis at an angle $\pi/6$. Find the points. Also, find the points on the $x-$ axis at which the tangent lines hit.

2. Solution

Implicit differentiation is only a part of this problem. First, we have to determine an approach for our solution. What does is mean for a line to intersect the $x-$ axis at an angle $\pi/6$? If you sketch of a line and then form a triangle by drawing a perpindicular line segment from a point on the line to the $x-$ axis, you should see that the slope of the line is the tangent of the angle the line makes with the $x-$ axis.



Since we are interested only in the magnitude of the angle (not the sign), this means that we want to find points on the ellipse where the tangent line has slope $\pm\tan(\pi/6) = \pm 1/\sqrt{3}$. Of course, the slope of the tangent line is given by the value of the derivative of a function. But in this case, our equation is

$$\frac{x^2}{4} + y^2 = 1,$$

which is not the equation of a function. We could solve for y :

$$y = \pm\sqrt{1 - \frac{x^2}{4}}.$$

Choosing the positive root would correspond to the upper half of the ellipse, while the negative root would represent the lower half. But, we can also use implicit differentiation. If we differentiate both sides of the equation for the ellipse, thinking of y as a function of x, we obtain

$$\begin{aligned}\frac{2x}{4} + 2y\,\frac{dy}{dx} &= 0, \text{ or} \\ \frac{dy}{dx} &= -\frac{x}{4y}.\end{aligned}$$

The fact that y remains in the equation makes sense; the choice of a point on the upper or lower half of the ellipse will alter the sign, as we expect from looking at the picture. (The slopes of the line on the upper and lower halves of the ellipse should have opposite signs.)

Let's think about the portion of the ellipse in the first quadrant. The tangent lines at points in this quadrant will have negative slope, so we will be looking for a point on the ellipse with

$$\begin{aligned}-\frac{x}{4y} &= -\frac{1}{\sqrt{3}}, \text{ so} \\ \sqrt{3}x &= 4y \text{ or} \\ x &= \frac{4y}{\sqrt{3}}.\end{aligned}$$

Now we must substitute this into the equation of the ellipse.

$$\begin{aligned}\frac{\left(4y/\sqrt{3}\right)^2}{4} + y^2 &= 1 \\ \frac{4y^2}{3} + y^2 &= 1 \\ \frac{7}{3}y^2 &= 1 \\ y^2 &= \frac{3}{7} \\ y &= \sqrt{\frac{3}{7}},\end{aligned}$$

where we choose the positive root because we are looking for a point in the first quadrant. Also,

$$\begin{aligned}x &= \frac{4y}{\sqrt{3}} = \frac{4}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{7}} \\ &= \frac{4}{\sqrt{7}}.\end{aligned}$$

So, the point in the first quadrant at which the tangent line has slope $-1/\sqrt{3}$ is $\left(4/\sqrt{7}, \sqrt{3/7}\right)$. (By the way, at this point it might be a good idea to check that this point really is on the ellipse and that the formula for dy/dx really does give $-1/\sqrt{3}$ at this point. This serves as a check on your arithmetic.)

Now we could solve this problem again in each quadrant, but from the symmetry of the problem it is clear that the other points are related to this one by sign changes, so the 4 points will be

$$\left(\pm\frac{4}{\sqrt{7}},\pm\sqrt{\frac{3}{7}}\right).$$

Where will these lines intersect the $x-$ axis? Well, let's work with the first one: using the point-slope formula for a line, the equation will be

$$y-\sqrt{\frac{3}{7}}=-\frac{1}{\sqrt{3}}\left(x-\frac{4}{\sqrt{7}}\right).$$

This line will intersect the $x-$ axis when $y=0$, so

$$\begin{aligned}-\sqrt{\frac{3}{7}} &= -\frac{1}{\sqrt{3}}\left(x-\frac{4}{\sqrt{7}}\right)\\ \frac{3}{\sqrt{7}} &= x-\frac{4}{\sqrt{7}}\\ x &= \frac{7}{\sqrt{7}}=\sqrt{7}.\end{aligned}$$

Again, from the symmetry, two of the lines (the tangent lines in the first and fourth quadrants) will intersect the $x-$ axis at the point $\left(\sqrt{7},0\right)$ while the other two (second and third quadrants) will cross at $\left(-\sqrt{7},0\right)$.