Volumes of solids of revolution

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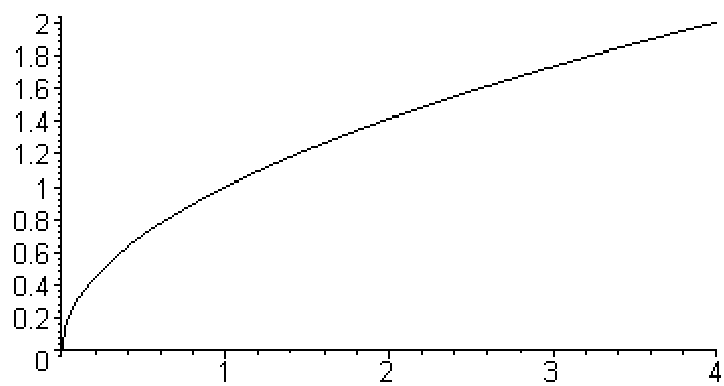
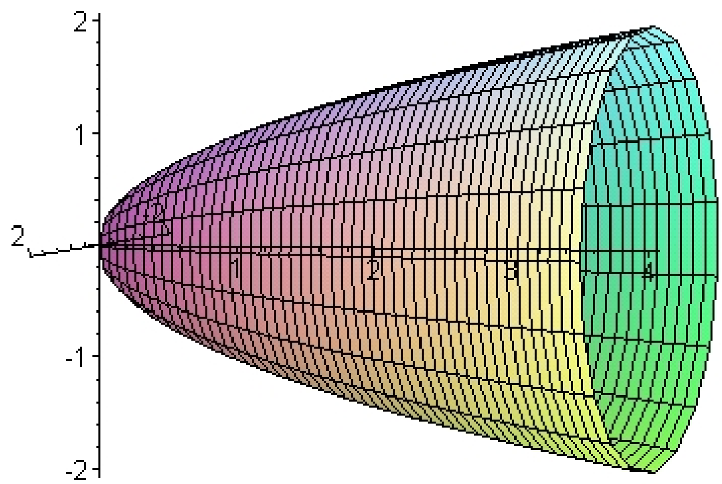
1. Volume of a section of a parabolic solid

In this example, we will compute the volume of a solid of revolution formed by revolving the graph of the square root about the $x-$ axis. Here is a graph showing the function on the interval $[0, 4]$.

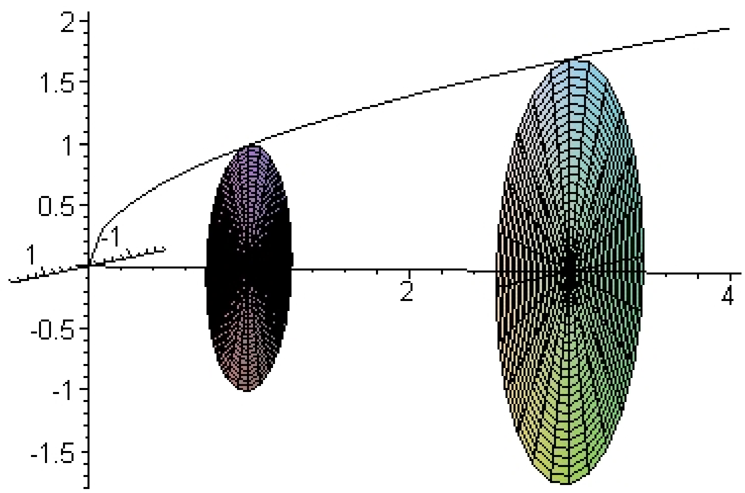


Note that, if $y = \sqrt{x}$ then $y^2 = x$; so the graph of the square root is simply one half of a parabola. A paraboloid is simply the surface formed by revolving a parabola about its axis of symmetry. Here is a plot showing the paraboloid generated by revolving the above section of the graph of the square root function about the $x-$ axis.



We want to compute the volume of the solid generated by rotating the region bounded by $y = \sqrt{x}$ and the $x-$ axis on the interval $[0, 4]$ about the $x-$ axis. That is, we want to compute the volume of the "cup" in the picture above.

What does a cross section of the solid look like? If we slice the solid perpindicular to the $x-$ axis, the cross section is a circle. Here is a plot showing the graph of $y = \sqrt{x}$ together with the cross sections at 0 and $\sqrt{3}$.



As you can see, the radius of the circles depends on where we have sliced the solid. In particular, if we slice the solid at x, the radius $r(x)$ of the circle will be $r(x) = \sqrt{x}$ (i.e. the y value on the graph of the square root function corresponding to x). The area $A(x)$ of the circle will be $A(x) = \pi\,(r\,(x))^2 = \pi\,(\sqrt{x})^2 = \pi x$. We want to "add up" the areas of all these circles in order to obtain the volume of the solid. That is,

$$\begin{aligned} Vol &= \int_0^4 A(x)\,dx \\ &= \int_0^4 \pi x\,dx \end{aligned}$$

$$
\begin{aligned}
&= \pi \left.\frac{x^2}{2}\right|_0^4 \\
&= \pi\left(\frac{16}{2}\right) = 8\pi.
\end{aligned}
$$

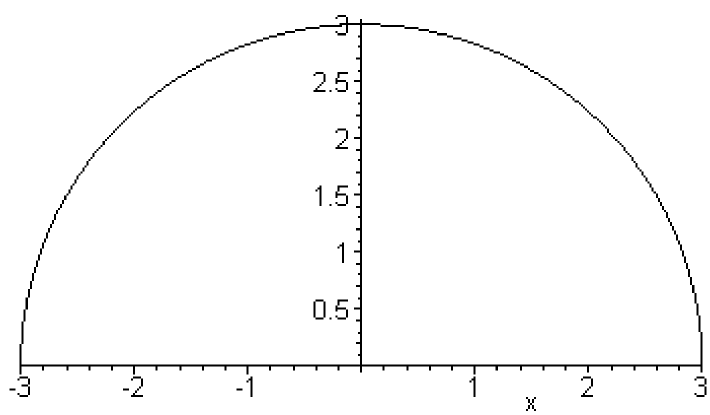
2. Volume of a sphere

You may recall from your previous math classes that the volume V of a sphere of radius r is

$$V = \frac{4}{3}\pi r^3.$$

We can now derive that formula using integration.

Here is a plot showing the graph of $y = \sqrt{9 - x^2}$ on the interval $[-3, 3]$.



Of course, this is simply the upper half of the circle of radius 3 centered at the origin. If we revolve this solid about the $x-$ axis, we will generate a sphere of radius 3. We should be able to compute the volume of the sphere by "adding up" the areas of the cross sections. The area $A(x)$ at a cross section is simply the area of a circle of radius $\sqrt{9 - x^2}$, that is, $A(x) = \pi\left(\sqrt{9 - x^2}\right)^2 = \pi\left(9 - x^2\right)$. The volume V of the sphere of radius 3 will then be

$$V = \int_{-3}^{3} \pi\left(9 - x^2\right)\, dx$$

$$
\begin{aligned}
&= 2\pi \int_0^3 \left(9 - x^2\right)\, dx \\
&= 2\pi \left(9x - \frac{x^3}{3}\right)\Bigg|_0^3 \\
&= 2\pi \left(27 - \frac{27}{3}\right) \\
&= 4\pi \left(\frac{27}{3}\right).
\end{aligned}
$$

(We left the answer in this form in order to compare it more easily to the formula for the volume of a sphere.)

The case for a sphere of radius r is not much different. In that case, we are rotating the graph of $y = \sqrt{r^2 - x^2}$ about the $x-$ axis on the interval $[-r, r]$. The cross sectional area $A(x) = \pi\left(r^2 - x^2\right)$ in this case, and the volume will be

$$
\begin{aligned}
V &= \int_{-r}^{r} \pi\left(r^2 - x^2\right)\, dx \\
&= 2\pi \int_0^r \left(r^2 - x^2\right)\, dx \\
&= 2\pi \left(r^2 x - \frac{x^3}{3}\right)\Bigg|_0^r \\
&= 2\pi \left(r^3 - \frac{r^3}{3}\right) \\
&= \frac{4}{3}\pi r^3.
\end{aligned}
$$