Integration by Parts

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Integration by parts is an extremely useful tool, both for computation and theory, in mathematical analysis. The basic formula

$$\int u\,dv = uv - \int v\,du$$

is simple enough. But there are lots of ways that it may be applied. Again, the best way to "learn" integration by parts is to work a lot of problems. The text portion of this site that covers integration by parts has some. Here are just a few more examples to help get you started.

1. Compute

$$\int x^5 \ln x\,dx$$

This problem is very easy once we realize that we should use integration by parts. There really isn't a choice for "u" and "v" since, right off, we don't know an antiderivative for $\ln x$. That pretty much forces us to choose

$$u = \ln x \text{ and } dv = x^5\,dx$$

so that

$$du = \frac{1}{x}dx \text{ and } v = \frac{x^6}{6}.$$

Now apply the integration by parts formula to obtain

$$\begin{aligned} \int x^5 \ln x\,dx &= \frac{x^6 \ln x}{6} - \int \frac{x^6}{6}\frac{1}{x}\,dx \\ &= \frac{x^6 \ln x}{6} - \int \frac{x^5}{6}\,dx \\ &= \frac{x^6 \ln x}{6} - \frac{x^6}{36} + c \end{aligned}$$

Obviously, $x^n \ln x$ could be handled in the same way for positive powers of x. In fact, you should check that negative powers work, too.

2. Powers of x times sines, cosines and exponentials can all be handled using integration by parts, though higher powers generally require repeated application. For example, consider

$$\int x^3 e^{2x}\,dx$$

Since differentiating or integrating an exponential function yields another exponential function (and the same for the sine and cosines), we will choose

$$u = x^3 \text{ and } dv = e^{2x}\,dx$$

so that an application of the integration by parts formula will reduce the power of x. Here it is:

$$du = 3x^2\,dx \text{ and } v = \frac{e^{2x}}{2}$$

so

$$\begin{aligned}\int x^3 e^{2x}\,dx &= \frac{x^3 e^{2x}}{2} - \int \frac{e^{2x}}{2}\left(3x^2\right)\,dx \\ &= \frac{x^3 e^{2x}}{2} - \frac{3}{2}\int x^2 e^{2x}\,dx\end{aligned}$$

Note that the remaining integral is similar to what we started with, but the power of x has been reduced. We can continue this until we have completely eliminated the power of x ("repeated integration by parts") to obtain

$$\int x^3 e^{2x}\,dx = \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{4}x e^{2x} - \frac{3}{8}e^{2x} + c$$

3. Here's one very similar to the last:

$$\int x^3 \cos 2x\,dx$$

Letting

$$u = x^3 \text{ and } dv = \cos 2x\,dx$$

a single application of integration by parts yields

$$\int x^3 \cos 2x\,dx = \frac{x^3 \sin 2x}{2} - \int \frac{\sin 2x}{2}\left(3x^2\right)\,dx$$

Note the similarity with the last problem. Continuing, we find

$$\int x^3 \cos 2x\,dx = \frac{1}{2}x^3 \sin 2x + \frac{3}{4}x^2 \cos 2x - \frac{3}{4}x \sin 2x - \frac{3}{8}\cos 2x + c$$

Compare this answer to the result for the previous example. Can you explain the similarities and the differences?

4. One last application of integration by parts that shows up quite frequently: the product of a sine or cosine with an exponential. In this case, the technique is used a little differently. We'll illustrate with an example.

$$\int e^{3x} \cos 5x\,dx$$

Let

$$u = e^{3x} \text{ and } dv = \cos 5x\,dx$$

so that

$$du = 3e^{3x}\,dx \text{ and } v = \frac{\sin 5x}{5}.$$

One application of the integration by parts formula yields

$$\int e^{3x}\cos 5x\,dx = \frac{e^{3x}\sin 5x}{5} - \frac{3}{5}\int e^{3x}\sin 5x\,dx$$

and, since the integral we have left is so similar to the one we started with, it doesn't look like we've made any progress. But, apply integration by parts again, this time with

$$u = e^{3x} \text{ and } dv = \sin 5x\,dx$$

Then we find

$$\begin{aligned}\int e^{3x}\cos 5x\,dx &= \frac{e^{3x}\sin 5x}{5} - \frac{3}{5}\int e^{3x}\sin 5x\,dx \\ &= \frac{e^{3x}\sin 5x}{5} - \frac{3}{5}\left(-\frac{e^{3x}\cos 5x}{5} - \int\left(-\frac{\cos 5x}{5}\right)\left(3e^{3x}\right)\,dx\right) \\ &= \frac{e^{3x}\sin 5x}{5} + \frac{3}{25}\,e^{3x}\cos 5x - \frac{9}{25}\int e^{3x}\cos 5x\,dx\end{aligned}$$

Now the integral that's left is identical to the one we started with, but that's not a bad sign in this case, since we can add that integral to both sides and obtain

$$\frac{34}{25}\int e^{3x}\cos 5x\,dx = \frac{e^{3x}\sin 5x}{5} + \frac{3}{25}\,e^{3x}\cos 5x$$

or

$$\begin{aligned}\int e^{3x}\cos 5x\,dx &= \frac{25}{34}\left(\frac{e^{3x}\sin 5x}{5} + \frac{3}{25}\,e^{3x}\cos 5x\right) \\ &= \frac{5}{34}e^{3x}\sin 5x + \frac{3}{34}e^{3x}\cos 5x\end{aligned}$$

Oh, I suppose we should add a constant of integration somewhere: so,

$$\int e^{3x}\cos 5x\,dx = \frac{5}{34}e^{3x}\sin 5x + \frac{3}{34}e^{3x}\cos 5x + c$$